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Classification of Directed and Hybrid Triple Systems

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Abstract

Pairwise nonisomorphic directed and hybrid triple systems can be generated by, respectively, directing and ordering twofold triple systems and using the automorphisms of the twofold triple systems for isomorph rejection. Using this approach directed triple systems of order up to 10 and hybrid triple systems of order up to 9 are classified. In particular, it turns out that the number of nonisomorphic directed triple systems of orders 9 and 10 are 596,893,386 and 3,753,619,614,456, respectively.

1 Introduction

A *triple system* of order v and index λ is a pair $(V, \mathcal{B})$ where V is a set with v elements, called *points*, and $\mathcal{B}$ is a collection of 3-element subsets of V, called *blocks*, such that every 2-element subset of points occurs in exactly λ blocks. In this paper, triple systems with $\lambda = 2$, called *twofold triple systems* (TTS for short, or $\mathrm{TTS}(v)$ to indicate the number of points), are utilized. The main focus of the study, however, is on certain triple systems with *ordered* blocks.

A *transitive triple* (a, b, c) is said to contain the ordered pairs (a, b), (a, c), and (b, c), whereas a *cyclic triple* $\langle a, b, c\rangle$ contains the ordered pairs (a, b), (b, c), and (c, a). Note that—when expressing a triple as a set of the ordered pairs it contains—$\langle a, b, c\rangle = \langle b, c, a\rangle = \langle c, a, b\rangle$, but no similar relation holds for transitive triples. Any three distinct points can obviously form six different transitive triples and two different cyclic triples.

We get three particular types of designs related to triple systems by considering transitive triples, cyclic triples, or both. A *directed triple system* of order v and index λ, $\mathrm{DTS}(v,\lambda)$ for short, is a pair $(\mathcal{V},\mathcal{D})$, where $\mathcal{D}$ is a collection of transitive triples on $\mathcal{V}$, a set of points of size v, so that each ordered pair of distinct points is contained in exactly λ triples [1, 7]. Directed triple systems are sometimes called transitive triple systems.

If we modify the definition of directed triple systems and use cyclic triples instead of transitive triples, we get the definition of a *Mendelsohn triple system*, $\mathrm{MTS}(v,\lambda)$ [7, 19]. Finally, if *both* cyclic triples and directed triples are allowed, we get the definition of a *hybrid triple system*, $\mathrm{HTS}(v,\lambda)$ [6, 7].

In this paper, we consider classification directed and hybrid triple systems with $\lambda = 1$, and mention Mendelsohn triple systems only occasionally. Whenever $\lambda = 1$, we omit λ from the notations and write $\mathrm{DTS}(v)$, $\mathrm{MTS}(v)$, and $\mathrm{HTS}(v)$. It is assumed that $v \geq 3$.

Some basic results on the designs under consideration are discussed in Section 2. In Section 3 classification of directed and hybrid triple systems via twofold triple systems is considered, and the classification results obtained are summarized in Section 4. Directed triple systems are classified up to order 10 and hybrid triple systems up to order 9. Finally, in Section 5, a clique approach is used to study orientability of twofold triple systems of order at most 10.

2 Preliminaries

By disregarding the ordering of the triples of an $\mathrm{HTS}(v,\lambda)$ ($\mathrm{DTS}(v,\lambda)$, $\mathrm{MTS}(v,\lambda)$), one obtains a triple system of order v and index 2λ, called the *underlying triple system*. In particular, each $\mathrm{HTS}(v)$ ($\mathrm{DTS}(v)$, $\mathrm{MTS}(v)$) has an underlying TTS. Several ways of proving the following theorem can be found in [7]; the result was proved by Colbourn and Colbourn [3] for $\lambda = 2$ and by Colbourn and Harms [5] for any even λ.

Theorem 1. *Every triple system with even index is the underlying triple system of some DTS (and therefore also some HTS).*

In particular, every $\mathrm{TTS}(v)$ is the underlying triple system of some $\mathrm{DTS}(v)$ (and therefore also some $\mathrm{HTS}(v)$).

Two HTSs (DTSs, MTSs, TTSs) are *isomorphic* if there exists a bijection between their point sets that also maps the triples of one system to the triples of the other (expressing transitive and cyclic triples as sets of the ordered pairs they

contain). Such a bijection is called an *isomorphism*. Isomorphisms from an HTS (DTS, MTS, TTS) onto itself are called *automorphisms* and form a group.
Converses of transitive and cyclic triples are defined as $(a,b,c)^{\mathcal{R}} = (c,b,a)$ and $\langle a,b,c\rangle^{\mathcal{R}} = \langle c,b,a\rangle$, respectively. The converse of an HTS (DTS, MTS)—obtained by taking the converse of all its triples—is also an HTS (DTS, MTS). Formally, the converse of $(\mathcal{V},\mathcal{D})$ is $(\mathcal{V},\mathcal{D}^{\mathcal{R}})$, where $\mathcal{D}^{\mathcal{R}} = \{B^{\mathcal{R}} : B \in \mathcal{D}\}$. An HTS (DTS, MTS) that is isomorphic to its converse is called *self-converse*.
Let, respectively, $D(v)$, $M(v)$, $H(v)$, and $T(v)$ denote the number of nonisomorphic directed, Mendelsohn, hybrid, and twofold triple systems of order v.

Theorem 2. *For* $v \equiv 0,1 \pmod 3$ *sufficiently large, the asymptotic growth rate of* $D(v)$*,* $M(v)$*,* $H(v)$*, and* $T(v)$ *is* $\exp(\frac{v^2 \ln v}{3}(1+o(1)))$.

Proof. Phelps and Lindner [26] determined the asymptotic growth rate for $D(v)$, $M(v)$, and $T(v)$; see also [7].
Since every DTS is an HTS, $H(v) \geq D(v) = \exp(\frac{v^2 \ln v}{3}(1+o(1)))$. As will be seen in Section 3, a $\mathrm{TTS}(v)$ underlies at most $2^{v(v-1)/2}$ $\mathrm{HTS}(v)$. Hence

$$\begin{aligned} H(v) &\leq 2^{v(v-1)/2} \exp(\frac{v^2 \ln v}{3}(1+o(1))) \\ &= \exp(\frac{v^2 \ln v}{3}(1+o(1)) + \ln 2^{v^2/2-v/2}) \\ &= \exp(\frac{v^2 \ln v}{3}(1+o(1) + \frac{3\ln 2}{2\ln v} - \frac{3\ln 2}{2v\ln v})) \\ &= \exp(\frac{v^2 \ln v}{3}(1+o(1))). \end{aligned}$$

□

Note that Theorem 2 gives the asymptotic growth rate in a rather broad form, and the values of the functions may differ even exponentially.
The previously known and new (in bold) exact values of $T(v)$, $D(v)$, and $H(v)$ are shown in Table 1, where empty entries indicate open cases. Exact values of $T(v)$ have been obtained in [22] for $v = 6$, in [23] for $v = 7$, in [21] (incorrect, one missing) and [17] for $v = 9$, in [4, 14] for $v = 10$, and in [25] for $v = 12$. The values of $D(v)$ for $v \leq 7$ have been obtained in [8]. As far as we know, no values of $H(v)$ have been published earlier (but determining $H(v)$ for $v \leq 4$ is very easy, so at least these values have probably been known).
As for MTSs, the number of inequivalent, rather than nonisomorphic, designs have been tabulated in the literature, two MTSs being equivalent if they are isomorphic

v	$T(v)$	$D(v)$	$H(v)$
3	1	1	$\mathbf{2}$
4	1	3	$\mathbf{7}$
6	1	32	$\mathbf{560}$
7	4	2,368	$\mathbf{100{,}204}$
9	36	$\mathbf{596{,}893{,}386}$	$\mathbf{546{,}282{,}250{,}538}$
10	960	$\mathbf{3{,}753{,}619{,}614{,}456}$	
12	242,995,846		
13			

Table 1: Number of designs

or if one is isomorphic to the converse of the other. The number of inequivalent MTSs for $v = 3$, 4, 6, 7, 9, 10, and 12 is, respectively, 1, 1, 0, 3, 18, 143 (the value 144 from [10], occasionally referenced in the literature, is incorrect), and 4,905,693; these values have been determined in [20] for $n \leq 7$, in [17] for $n = 9$, and in [9] for $n = 10, 12$.

There are two general frameworks for classifying designs of the types discussed here: direct construction or a classification via the twofold triple systems. In this paper, constructions of the latter type are considered. Since the number of inequivalent (alternatively, nonisomorphic) Mendelsohn triple systems is, for the known cases, smaller than the number of twofold triple systems, it seems that direct constructions should be preferred for those (see, however, the approach in [9]); in a separate study, an attempt will be made to classify the Mendelsohn triple systems of order 13.

3 Directing and Ordering Twofold Triple Systems

The idea of classifying designs of the types considered here via twofold triple systems is not new. For example, Mathon and Rosa [17] classified Mendelsohn triple systems of order 9 in this manner (but see the comments at the end of the previous section). The process of turning the blocks of twofold triple systems into transitive triples, cyclic triples, or both, is called *directing*, *orienting*, and *ordering*, respectively. From now on, only directed and hybrid triple systems are considered. Note that each pair of repeated triples of a TTS is unaffected by the directing and orienting of any other triple and may therefore be considered separately. There are three possible ways of directing such a pair of triples and one way of orienting it: $\{(a,b,c),(c,b,a)\}$, $\{(b,a,c),(c,a,b)\}$, and $\{(a,c,b),(b,c,a)\}$; and $\{\langle a,b,c\rangle,\langle c,b,a\rangle\}$, respectively.

3.1 Finding All DTSs by Directing

Theorem 1 tells that every TTS underlies a DTS. Several algorithms for obtaining a DTS having a given TTS as its underlying system have been published [12, 13]; see also [7]. However, none of those is useful here since they were designed primarily to find *one* DTS, not *all* DTSs, from the given underlying system. Our classification method relies on finding all ways to direct a given TTS via an instance of the exact cover problem. The NP-complete decision version of the exact cover problem is as follows:

Input: A collection of subsets $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ of a universal set U.
Question: Is there a subcollection $\mathcal{T} \subseteq \mathcal{S}$ that forms a partition of U?

It is perhaps somewhat counterintuitive that we employ an algorithm developed for an NP-complete problem when in fact the fastest known algorithm for finding *one* directing is linear in the number of blocks [13]. However, the problem of finding all solutions of such problem instances is fundamentally different from that of finding just one solution. The heuristic of the classic backtracking algorithm [11, 16] for the exact cover problem minimizes the branching factor on each level of the search tree; this is often a good heuristic for minimizing the size of the search tree for finding all solutions. It appears though that the types of instances considered in this study are quite insensitive to the choice of algorithm, with the number of nodes on each level of the search tree growing steadily.
An instance of the exact cover problem may be constructed as follows. Let the universal set U consist of the union of all (labeled) blocks of the TTS and all ordered pairs of distinct points. For each block $B = \{a,b,c\}$ of the TTS, six elements of $\mathcal{S}$ are now formed (corresponding to the six ways of directing a triple):

$$\begin{gathered}\{B,(a,b),(a,c),(b,c)\},\\ \{B,(c,b),(c,a),(b,a)\},\\ \{B,(b,a),(b,c),(a,c)\},\\ \{B,(c,a),(c,b),(a,b)\},\\ \{B,(a,c),(a,b),(c,b)\},\\ \{B,(b,c),(b,a),(c,a)\}.\end{gathered}$$

Recall that the repeated triples are considered separately and are therefore ignored (including the pairs of points they contain) in formulating and solving the exact cover instance. The overall algorithm, including the issue of isomorph rejection, will be postponed until we have discussed the procedure of obtaining hybrid triple systems by ordering.

3.2 Finding All HTSs by Ordering

Ordering a given TTS to arrive at an HTS is an easy task. Namely, if we view the TTS as a decomposition of the multigraph $2K_v$ into triangles, there are exactly two possibilities for directing any pair of edges with the same endpoints. If we distinguish between the blocks in a pair of multiple blocks, then the number of labeled HTSs is then $2^{v(v-1)/2}$. However, here we do not distinguish between the blocks in a pair of multiple blocks; then the number of labeled HTSs is

$$4^r \cdot 2^{v(v-1)/2-3r} = 2^{v(v-1)/2-r}, \tag{1}$$

where r is the number of pairs of multiple blocks.
Finding all these HTSs is no challenge from a computational point of view, and just counting the labeled such objects is straightforward. The fact that all labeled objects are easily counted implies that if we are merely interested in counting the nonisomorphic HTSs, this can be done by the orbit-stabilizer theorem if the objects with nontrivial automorphism group can be classified. Let N_i be the number of nonisomorphic objects with prescribed parameters whose automorphism group has order i. By the orbit-stabilizer theorem, the total number of labeled objects is

$$N = v! \sum_i \frac{N_i}{i}. \tag{2}$$

Consequently, it seems feasible to extend the column $H(v)$ of Table 1 along such an approach.
However, the focus of this study is on classifying objects to get explicit representatives from each isomorphism class. Isomorph rejection, to be considered next, plays a central role in any classification.

3.3 Isomorph Rejection

The following obvious result is the cornerstone in detecting isomorphisms between DTSs (HTSs).

Lemma 3. *Assume τ is an isomorphism between two DTSs (HTSs). Then τ is also an isomorphism between the respective underlying triple systems. In particular, if* *those DTSs (HTSs) have the same labeled underlying triple system, then τ is an automorphism.*

An implication of this lemma is that if we direct (order) two nonisomorphic TTSs, then we know that the DTSs (HTSs) we get are nonisomorphic. Moreover, since an isomorphism between two DTSs (HTSs) encountered in the search is an automorphism of the TTS considered, it suffices to consider elements of the automorphism group of this TTS to detect such isomorphisms. This also implies that if a TTS has no nontrivial automorphisms, then the DTSs (HTSs) one gets are necessarily nonisomorphic.

Our isomorph rejection procedure checks whether a DTS (HTS) found is the smallest (under a prescribed total order of labeled designs) in its orbit under the automorphism group of the TTS considered; see also [9, p. 249]. If that is the case, it is accepted. As mentioned earlier, this means that if the automorphism group of the TTS is trivial, then a DTS (HTS) is always accepted. In performing this test, we simultaneously get the automorphism group of the DTS (HTS); in this work we have collected the orders of these groups.

The outlined test can also be used to check whether a DTS (HTS) is self-converse. Namely, this is the case if and only if its converse is in the orbit of the DTS (HTS) under the automorphism group of the TTS considered. Note that if a TTS consists only of pairs of repeated blocks, then any DTS (HTS) found is self-converse.

For the parameters considered in this work, $v \leq 10$, there are very few nonisomorphic TTS, and determining the automorphism groups of these is rather straightforward, for example, using the program *nauty* [18]. Algorithms for handling permutation groups can be found in [2].

3.4 Overall Algorithm and Validation

We now need to put together the three subtasks specified so far: directing (ordering) of repeated blocks, directing (ordering) of blocks that are not repeated, and isomorph rejection.

Note that an automorphism always maps repeated blocks among themselves and the same holds for blocks that are not repeated. Based on this observation, the following approach may be taken. For a given TTS, find all possible directings (orderings) of the blocks that are not repeated. For each directing (ordering) found, carry out isomorph rejection as outlined in the previous subsection. If the partial design is accepted, direct (order) the repeated blocks, and carry out isomorph rejection for those. Observe that in the final step only automorphisms of the partial design are considered in the isomorph rejection procedure.

In any computer-aided classification, the validity and reliability of the results should be addressed. Here a double-counting technique utilizing the orbit-stabilizer theorem (2) is employed. Analogous techniques have earlier been used, for example, in [15].

Let $\mathscr{D}$ denote the set of all nonisomorphic $\mathrm{DTS}(v)$. By the orbit-stabilizer theorem, the total number of labeled $\mathrm{DTS}(v)$ is

$$v! \sum_{\mathcal{D} \in \mathscr{D}} \frac{1}{\mathrm{Aut}(\mathcal{D})}. \tag{3}$$

Similarly, the total number of labeled $\mathrm{DTS}(v)$ may also be calculated as

$$v! \sum_{\mathcal{B} \in \mathscr{T}} \frac{L(\mathcal{B})}{\mathrm{Aut}(\mathcal{B})}, \tag{4}$$

where $\mathscr{B}$ is the set of nonisomorphic $\mathrm{TTS}(v)$ and $L(\mathcal{B})$ is the total number of labeled DTSs having $\mathcal{B}$ as the underlying triple system. Data for (3) is obtained from the accepted DTSs, whereas data for (4) comes from all directings encountered during the search. Note that some care must be taken to get the correct values for (4) if the outlined two-step method is used, where partial directings may be rejected.

One drawback of double-counting directings in the described manner is that in directing a TTS with a trivial automorphism group, a loss of a DTS will be left unnoticed. With this technique, there is a trade-off between time and quality: if an object occurs many times during the search, this is good for validation but bad for the overall computation time.

The formulas for the number of HTSs are analogous to (3) and (4). By (1), however, (4) reduces to

$$v! \sum_{\mathcal{B} \in \mathscr{B}} \frac{2^{v(v-1)/2 - R(\mathcal{B})}}{\mathrm{Aut}(\mathcal{B})}, \tag{5}$$

where $R(\mathcal{B})$ is the number of pairs of multiple blocks in $\mathcal{B}$, giving a much higher level of confidence for the validity check in this case.

4 The Results

In this section, the main results are presented along with some additional comments. The recorded CPU times apply to the actual computer runs carried out using (initial) algorithms that differ slightly from those outlined in the main text. The

CPU times should therefore be viewed as indications of the hardness of instances (or upper bounds on the CPU times needed), rather than measures of optimized algorithms.

4.1 The Case $\mathrm{DTS}(9)$

The 36 nonisomorphic $\mathrm{TTS}(9)$ lead to 596,893,386 pairwise nonisomorphic directed triple systems of order 9, out of which 37,172 are self-converse and 131,643 have a nontrivial automorphism group. The total number of labeled DTS(9) is 216,576,683,027,712.

A list of open problems in [7] begins by asking how many (how few) nonisomorphic $\mathrm{DTS}(v,\lambda)$ a given TS$(v, 2\lambda)$ can underlie. More specifically, and relevantly, some twofold triple systems underlie exponentially many (with respect to v) DTS, but does *every* TTS? The results (of this and other studies) seem to indicate that the answer is yes.

The minimum and maximum number of nonisomorphic DTSs obtained from a $\mathrm{TTS}(9)$ are 1,263 and 67,826,496, respectively.

Slightly over 7 hours of CPU time of a 1.3-GHz PC was used in this search. Data regarding the orders of the automorphism groups and the number of self-converse designs are tabulated in Table 2.

$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	number of DTS	number of self-converse DTS
1	596,761,743	36,481
2	130,251	625
3	900	20
4	366	38
5	2	0
6	94	2
8	17	3
10	3	1
16	6	2
20	4	0
Total	596,893,386	37,172

Table 2: Directed triple systems of order 9

4.2 The Case $\mathrm{DTS}(10)$

The 960 nonisomorphic $\mathrm{TTS}(10)$ lead to 3,753,619,614,456 pairwise nonisomorphic directed triple systems of order 10, of which 664,140 are self-converse and 84,263 have a nontrivial automorphism group. The total number of labeled $\mathrm{DTS}(10)$ is 13,621,134,683,949,096,960.
The minimum and maximum number of nonisomorphic DTSs obtained from a $\mathrm{TTS}(10)$ are 1,063,665 and 6,527,003,056, respectively.
This classification required almost 16 months of CPU time using mostly 2.2–2.4-Ghz PCs but also a couple of slower computers. Data regarding the orders of the automorphism groups and the number of self-converse designs are tabulated in Table 3.

$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	number of DTS	number of self-converse DTS
1	3,753,619,530,193	664,057
2	51,186	40
3	32,883	43
5	176	0
6	6	0
7	12	0
Total	3,753,619,614,456	664,140

Table 3: Directed triple systems of order 10

4.3 The Cases $\mathrm{HTS}(v)$, $v \leq 9$

As far as we know, no classification of hybrid triple systems has earlier been published, so we present the classification results for all admissible cases $3 \leq v \leq 9$. The results are compiled in Tables 4 through 8.
The HTSs of order 3 can be classified even without a computer. There are two nonisomorphic systems: one directed system and one Mendelsohn system, both of which are self-converse and have a nontrivial automorphism group. The total number of labeled $\mathrm{HTS}(3)$ is 4.
The unique $\mathrm{TTS}(4)$ leads to 7 nonisomorphic hybrid triple systems of order 4, of which 5 are self-converse. All except one have a nontrivial automorphism group. The total number of labeled $\mathrm{HTS}(4)$ is 64.
The unique $\mathrm{TTS}(6)$ leads to 560 hybrid triple systems of order 6, of which 20 have a nontrivial automorphism group. The case $v = 6$ is special in at least two ways:

none of the $\mathrm{HTS}(6)$ is self-converse and no Mendelsohn triple system of order 6 exists. The total number of labeled $\mathrm{HTS}(6)$ is $393,216$.

The four nonisomorphic $\mathrm{TTS}(7)$ lead to 100,204 nonisomorphic hybrid triple systems of order 7, of which 1,508 are self-converse and $1,953$ have a nontrivial automorphism group. The minimum and maximum number of nonisomorphic HTSs obtained from a $\mathrm{TTS}(7)$ are 140 and 50,664, respectively. The total number of labeled $\mathrm{HTS}(7)$ is 499,875,840.

The 36 nonisomorphic $\mathrm{TTS}(9)$ lead to 546,282,250,538 pairwise nonisomorphic hybrid systems of order 9, of which 9,126,074 are self-converse and 4,271,259 have a nontrivial automorphism group. The minimum and maximum number of nonisomorphic HTSs obtained from a $\mathrm{TTS}(9)$ are 39,662 and 68,719,476,736 (=2^{36}), respectively. The total number of labeled $\mathrm{HTS}(9)$ is 198,234,126,560, 526,336.

Approximately two months of CPU time was used in the classification of $\mathrm{HTS}(9)$, utilizing 1.2–2.4-GHz PCs. Less than one minute of CPU time was enough for classifying all hybrid systems of smaller orders.

$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	number of HTS	number of self-converse HTS
2	1	1
6	1	1
Total	2	2

Table 4: Hybrid triple systems of order 3

$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	number of HTS	number of self-converse HTS
1	1	1
2	1	1
3	1	1
4	3	1
12	1	1
Total	7	5

Table 5: Hybrid triple systems of order 4

$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	number of HTS	number of self-converse HTS
1	540	0
3	16	0
5	4	0
Total	560	0

Table 6: Hybrid triple systems of order 6

$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	number of HTS	number of self-converse HTS
1	98,251	1,247
2	1,759	185
3	102	10
4	42	42
6	29	7
7	1	1
8	11	11
14	2	0
24	4	4
42	2	0
168	1	1
Total	100,204	1,508

Table 7: Hybrid triple systems of order 7

$\lvert \mathrm{Aut}(\mathcal{D}) \rvert$	number of HTS	number of self-converse HTS
1	546,277,979,279	9,094,777
2	4,248,194	28,964
3	19,244	1,610
4	2,974	546
5	240	0
6	428	106
8	82	22
9	9	1
10	28	4
12	31	31
16	14	6
18	3	3
20	8	0
36	1	1
48	1	1
54	1	1
432	1	1
Total	546,282,250,538	9,126,074

Table 8: Hybrid triple systems of order 9

5 Orientability of Twofold Triple Systems

By Theorem 1, every TTS can be directed. However, not every TTS can be oriented—see the classification results listed in Section 2. For TTSs that cannot be oriented, it is interesting to study the maximum number of triples that can be oriented (in precise terms: the maximum number of cyclic triples an HTS with this underlying TTS can have). The minimum over all $\mathrm{TTS}(v)$ of this number is denoted by $mc(v)$.

The values of $mc(v)$ for the smallest v are known. From the classification results in Section 2 one can see that the unique $\mathrm{TTS}(3)$ and $\mathrm{TTS}(4)$ lead to $\mathrm{MTS}(3)$ and $\mathrm{MTS}(4)$, so $mc(3) = 2$ and $mc(4) = 4$. The unique $\mathrm{TTS}(6)$ cannot be oriented and $mc(6) = 7$ [6]. It is further known that $mc(7) = 11$ and $mc(9) = 18$ [6].

In the current classification of hybrid triple systems, the classified systems were checked for the number of cyclic triples. After all, one of the main motivations for classifying combinatorial objects is to get a complete set of objects, which can be checked for various properties. However, we shall here describe a direct and much more efficient approach for obtaining the same result using a clique approach.

From a given $\mathrm{TTS}(v)$ with $b = v(v-1)/3$ blocks, we construct a graph G of order $2b$. The graph G has two vertices for each triple of the TTS, one for each possible way of orienting it. Two vertices of G are adjacent exactly when the corresponding oriented blocks (as sets of ordered pairs) do not contain the same ordered pair. Two vertices originating from the same triple of the TTS are nonadjacent. The size of a maximum clique in this graph gives the maximum number of blocks that can be oriented.

Given a classification of $\mathrm{TTS}(v)$ for $v \leq 10$, it takes just seconds to determine $mc(v)$ for these parameters. Using the program Cliquer [24], we obtained $mc(10) = 24$ (the bound $mc(10) \geq 23$ was obtained in [6]).

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