The Steiner Quadruple Systems of Order 16

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Dedicated to the memory of Professor Jack van Lint

Abstract

The Steiner quadruple systems of order 16 are classified up to isomorphism by means of an exhaustive computer search. The number of isomorphism classes of such designs is 1,054,163. Properties of the designs—including the orders of the automorphism groups and the structures of the derived Steiner triple systems of order 15—are tabulated. A double-counting consistency check is carried out to gain confidence in the correctness of the classification.

Key words: Classification; Derived design; Steiner system; SQS

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1 Introduction

A standard problem in design theory is that of classifying designs with specific parameter values up to isomorphism—an example of such a classification is the one obtained by Van Lint, Van Tilborg, and Wiekema in [13].

For positive integers $2 \le t \le k \le v$, a *Steiner system* $S(t, k, v)$ is a v-set V of *points* together with a set of k-subsets of V, called *blocks*, with the property that every t-subset of points is contained in exactly one block. Steiner systems $S(2, 3, v)$ and $S(3, 4, v)$ are called *Steiner triple systems* and *Steiner quadruple systems*, respectively. The parameter v is called the *order* of a system. For the current knowledge as regards classification up to isomorphism of Steiner triple systems, see [8]. Writing $N(t, k, v)$ for the number of isomorphism classes of Steiner systems $S(t, k, v)$, for Steiner quadruple systems it is now known that

$$\begin{aligned} N(3,4,8) &= 1, \\ N(3,4,10) &= 1, \\ N(3,4,14) &= 4, \\ N(3,4,16) &= 1{,}054{,}163, \end{aligned}$$

obtained in [1], [1], [19], and the current work, respectively. See also [2,21]. It is well known [3] that $N(3, 4, v) > 0$ exactly when $v \equiv 2$ or $4 \pmod 6$.

Asymptotically it is known [4,10] that

$$cv^3 \le \log N(3,4,v) \le (v^3 \log v)/24$$

for some constant $c > 0$ and sufficiently large admissible v.

In the present paper the constructive enumeration that was carried out to determine $N(3, 4, 16) = 1{,}054{,}163$ is documented. This brings to an end work on improving lower bounds on $N(3, 4, 16)$ carried out by Doyen and Vandensavel [4]; Gibbons and Mathon [5]; Gibbons, Mathon, and Corneil [6]; Lindner and Rosa [11,12]; and Zinoviev and Zinoviev [22] over a span of more than three decades.

A Steiner system, as defined here, is a certain set system. Formally, a *set system* is a pair $\mathcal{X} = (V, \mathcal{B})$, where V is a finite set of *points* and $\mathcal{B}$ is a set of subsets (called *blocks*) of V.

An *isomorphism* of a set system $\mathcal{X} = (V, \mathcal{B})$ onto a set system $\mathcal{X}' = (V', \mathcal{B}')$ is a bijection $f : V \to V'$ satisfying $\mathcal{B}' = \{f(B) : B \in \mathcal{B}\}$. If such a bijection exists then we say that $\mathcal{X}$ and $\mathcal{X}'$ are *isomorphic*. An *automorphism* of $\mathcal{X}$ is an isomorphism of $\mathcal{X}$ onto itself. The set of all automorphisms of $\mathcal{X}$ with composition of permutations as the group operation forms the *automorphism group* of $\mathcal{X}$, denoted by $\mathrm{Aut}(\mathcal{X})$.

Let $\mathcal{Q} = (V, \mathcal{B})$ be a Steiner system $S(t, k, v)$, and let $x \in V$. The *derived system* of $\mathcal{Q}$ with respect to x is the set system $\mathcal{Q}_x = (V_x, \mathcal{B}_x)$ defined by $V_x = V \setminus \{x\}$ and $\mathcal{B}_x = \{B \setminus \{x\} : x \in B \in \mathcal{B}\}$. When $t \ge 3$, it is immediate that $\mathcal{Q}_x$ is a Steiner system $S(t - 1, k - 1, v - 1)$.

The paper is outlined as follows. The classification approach, divided into the parts of constructing designs and isomorph rejection, is considered in Section 2. A consistency check that was performed to gain confidence in the correctness of the classification is also discussed. The paper is concluded in Section 3 by studying and tabulating various properties of the classified designs, including the orders of the automorphism groups and the structures of the derived Steiner triple systems of order 15.

2 Classification

Because a Steiner quadruple system $S(3, 4, v)$ defines v derived Steiner triple systems $S(2, 3, v-1)$, a direct classification approach is to proceed by extending triple systems with a new point and new blocks to obtain all quadruple systems. Mendelsohn and Hung [19] used this basic approach to classify the Steiner quadruple systems $S(3, 4, 14)$, and this is the approach we employ here to classify the Steiner quadruple systems $S(3, 4, 16)$. In terms of the algorithms employed, the present approach is closely related to the approach used in [8] to classify the Steiner triple systems $S(2, 3, 19)$.

Let us now describe the approach in more detail. We start with one representative system from each of the 80 isomorphism classes of the Steiner triple systems $S(2, 3, 15)$; such representatives can be found, for example, in [15]. For brevity in what follows, we use the terms "triple system" and "quadruple system" exclusively to refer to a Steiner triple system $S(2, 3, 15)$ and a Steiner quadruple system $S(3, 4, 16)$, respectively.

Extending triple systems. For each representative triple system $\mathcal{T}$, we introduce a new point p to the point set, insert p into all the blocks, and extend the resulting set system $\hat{\mathcal{T}}$ into a quadruple system in all possible ways by adding blocks.

We implement the extension step by formulating the extension problem as an instance of the exact cover problem. In the exact cover problem one is given a set $\mathcal{C}$ of subsets of a finite set U, and the task is to produce all partitions of U consisting of sets in $\mathcal{C}$.

To extend $\hat{\mathcal{T}}$ into a quadruple system in all possible ways, we observe that

$$\mathcal{E} = \Big\{\{x, y, z, w\} \subset \hat{V} : |B \cap \{x, y, z, w\}| \leq 2 \text{ for all } B \in \hat{\mathcal{B}}\Big\}$$

consists of all canidate quadruples that can extend $\hat{\mathcal{T}}$, and let

$$U = \Big\{\{x, y, z\} \subset \hat{V} : \{x, y, z\} \not\subset B \text{ for all } B \in \hat{\mathcal{B}}\Big\},$$

$$\mathcal{C} = \{\{\{x, y, z\}, \{x, y, w\}, \{x, w, z\}, \{y, w, z\}\} : \{x, y, z, w\} \in \mathcal{E}\}.$$

To solve the instances of exact cover, we employ a backtrack algorithm of Knuth [9]. No isomorph rejection is carried out during the extension phase.

The extension from triple systems to quadruple systems required approximately 12 years of CPU time when distributed using the batch system `autoson`

[17] to a network of computers. A total of 107 computers, almost all with 2.2-GHz or 2.4-GHz Intel CPUs, were at some point running the search.

Isomorph rejection. The search yields a list of 325,895,777 quadruple systems extending the representative triple systems. To reject all but one quadruple system from every isomorphism class in this list, we employ two tests that are based on the general theory developed by McKay [18] and analogous to the tests in [8]. The first test ensures that any two isomorphic quadruple systems accepted in the test must be generated by extending the same representative triple system $\mathcal{T}$, and furthermore, must be related by an isomorphism in $\mathrm{Aut}(\hat{\mathcal{T}})$. The second test eliminates any remaining systems that are isomorphic under $\mathrm{Aut}(\hat{\mathcal{T}})$.

The first test is defined by means of a rule M that associates with every quadruple system $\mathcal{Q}$ a nonempty $\mathrm{Aut}(\mathcal{Q})$-orbit $M(\mathcal{Q})$ of points such that whenever $\mathcal{Q}'$ is a quadruple system isomorphic to $\mathcal{Q}$, every isomorphism of $\mathcal{Q}$ onto $\mathcal{Q}'$ maps $M(\mathcal{Q})$ onto $M(\mathcal{Q}')$. The first test accepts a quadruple system $\mathcal{Q}$ if $p \in M(\mathcal{Q})$, where p is the point of $\mathcal{Q}$ satisfying $\mathcal{Q}_p = \mathcal{T}$.

In the second test, we assume that the set of quadruple systems extending $\hat{\mathcal{T}}$ is ordered lexicographically and that $\mathrm{Aut}(\hat{\mathcal{T}})$ acts on this set by permuting the points in the blocks. The second test accepts $\mathcal{Q}$ if it is the lexicographic minimum of its $\mathrm{Aut}(\hat{\mathcal{T}})$-orbit.

Implementation of isomorph rejection. Efficient implementation of the two outlined tests is somewhat nontrivial, so we will discuss this issue here. For a given quadruple system $\mathcal{Q} = (V, \mathcal{B})$, each derived triple system $\mathcal{Q}_x$ belongs to one of the 80 isomorphism classes of triple systems. These isomorphism classes can be distinguished from each other by the multiset that, for each point y, consists of the number of occurrences of y in Pasch configurations; a *Pasch configuration* is a set of four triples isomorphic to $\{\{a, c, e\}, \{a, d, f\}, \{b, c, f\}, \{b, d, e\}\}$. We number the 80 isomorphism classes of triple systems so that the order of the automorphism group is a nondecreasing function of the numbering.

The first test is implemented as follows. Given a quadruple system $\mathcal{Q} = (V, \mathcal{B})$, we compute the index of every derived triple system $\mathcal{Q}_x$ using the Pasch configuration invariant. If the derived triple system $\mathcal{Q}_p$ does not have the minimum index over all derived triple systems, then we reject $\mathcal{Q}$ from further consideration. Otherwise we represent $\mathcal{Q}$ as the bipartite incidence graph with $V \cup \mathcal{B}$ as the vertex set and $\{\{x, B\} : x \in B \in \mathcal{B}\}$ as the edge set. We then compute a canonical labeling and automorphism orbits for the incidence graph of $\mathcal{Q}$ using *nauty* [16], and, in the canonically labeled incidence graph, let $M(\mathcal{Q})$ be the first automorphism orbit of points with the property that the derived triple systems associated with the points of the orbit have the minimum number over all derived triple systems. We then accept $\mathcal{Q}$ if $p \in M(\mathcal{Q})$. In invoking *nauty*, we use the numbers of the derived triple systems to define an initial ordered partition for expediting the evaluation of the canonical labeling. More precisely, points with identically numbered derived triple systems are placed into the same cell of the partition, and the cells are ordered by the numbers of the derived triple systems.

The second test is implemented as an exhaustive search over a precomputed table of elements of $\mathrm{Aut}(\hat{\mathcal{T}})$. If lexicographically $\mathcal{Q}^a < \mathcal{Q}$ for some $a \in \mathrm{Aut}(\hat{\mathcal{T}})$, then we reject $\mathcal{Q}$. Otherwise we accept $\mathcal{Q}$ as the representative of its isomorphism class.

When implemented in this manner, isomorph rejection for the generated quadruple systems requires less than a day of CPU time.

A consistency check. To gain confidence in the correctness of the classification, we perform a consistency check based on double counting. On one hand, we rely on data obtained in the search for the extensions of the triple systems; on the other hand, we rely on the classified quadruple systems—cf. [8].

From the search data we obtain for each representative triple system $\mathcal{T}$ the total number $E(\mathcal{T})$ of quadruple systems that extend $\hat{\mathcal{T}}$. Note that $\mathrm{Aut}(\mathcal{T})$ is isomorphic to $\mathrm{Aut}(\hat{\mathcal{T}})$. By the orbit-stabilizer theorem we obtain that the total number of quadruple systems over a fixed set of 16 points is

$$\frac{1}{16}\sum_{\mathcal{T}} \frac{16! \cdot E(\mathcal{T})}{|\mathrm{Aut}(\mathcal{T})|} = \sum_{\mathcal{T}} \frac{15! \cdot E(\mathcal{T})}{|\mathrm{Aut}(\mathcal{T})|},$$

where the division by 16 is required because the sum counts each quadruple system once for each point.

Looking at the classified quadruple systems and their automorphism groups, from the orbit-stabilizer theorem we obtain an alternative count

$$\sum_{\mathcal{Q}} \frac{16!}{|\mathrm{Aut}(\mathcal{Q})|},$$

where the sum is over the classified quadruple systems.

Both counts give the result 14,311,959,985,625,702,400, which gives us confidence that the classification is correct.

3 Results

In this section we collect some further results that have been computed from the classified designs; however, let us first state the main result.

Theorem 1 *The number of isomorphism classes of Steiner quadruple systems* $S(3,4,16)$ *is* 1,054,163.

Two properties that are obtained directly with the outlined method of classification are the automorphism group and the structures of the derived triple systems. Table 1 gives the number of isomorphism classes of quadruple systems for each possible order of the automorphism group.

INSERT TABLE 1 ABOUT HERE

For a given quadruple system $\mathcal{Q}$, denote by $\beta(\mathcal{Q})$ the number of isomorphism classes of derived triple systems of $\mathcal{Q}$. Table 2 partitions the isomorphism classes of quadruple systems based on the value of β.

INSERT TABLE 2 ABOUT HERE

If $\beta(\mathcal{Q}) = 1$, then $\mathcal{Q}$ is called *homogeneous*, and if $\beta(\mathcal{Q}) = v$, then $\mathcal{Q}$ is called *heterogeneous*. De Vries [20] has established that 69 of the 80 isomorphism classes of triple systems admit extension to a homogeneous quadruple system. Based on the classification we obtain that none of the remaining 11 isomorphism classes admits extension to a homogeneous quadruple system. Hartman and Phelps [7] conjecture that asymptotically "almost all" Steiner quadruple systems $S(3, 4, v)$ are heterogeneous.

Table 3 shows for each of the 80 isomorphism classes of derived triple systems the number of isomorphism classes of quadruple systems that extend triple systems in the class. The numbering of the derived triple systems is that used by Mathon, Phelps, and Rosa [15].

INSERT TABLE 3 ABOUT HERE

The *rank* of a quadruple system is the rank of a corresponding incidence matrix. Table 4 displays the number of isomorphism classes of quadruple systems for each possible rank. The results for rank at most 13 agree with those obtained by Zinoviev and Zinoviev [22].

INSERT TABLE 4 ABOUT HERE

Two further properties that we have investigated are colorability and resolvability.

A *weak k-coloring* of a design is a coloring of the points with the property that any block has at least two points colored with different colors and all k colors are used to color at least one point each. It is known that there exist Steiner quadruple systems $S(3, 4, 16)$ that are weakly 2-colorable and those that are not weakly 2-colorable [4]. Moreover, every Steiner quadruple system $S(3, 4, 16)$ is weakly 3-colorable [14]. Investigation of the classified designs shows that there are 349,058 weakly 2-colorable Steiner quadruple systems $S(3, 4, 16)$.

A design is *resolvable* if its blocks can be partitioned into *parallel classes*, which in turn partition the point set. Whereas for many types of designs the problem whether resolvable designs with given parameters exist or not is the focus of interest, for Steiner quadruple systems $S(3, 4, 16)$ the open problem has been whether designs that are *not* resolvable exist [22]. It turns out, however, that several hundred thousand of the Steiner quadruple systems $S(3, 4, 16)$ indeed are not resolvable.

Many properties remain to be studied. One of the most interesting—and, conceivably, computationally challenging—problems is that of determining whether the Steiner quadruple systems $S(3, 4, 16)$ can be extended to Steiner systems $S(4, 5, 17)$. In fact, the existence of Steiner systems $S(t, t+1, t+13)$ is an open problem for $4 \leq t \leq 11$ [3].

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Table 1
Orders of automorphism groups

$\lvert\mathrm{Aut}(\mathcal{Q})\rvert$	#	$\lvert\mathrm{Aut}(\mathcal{Q})\rvert$	#	$\lvert\mathrm{Aut}(\mathcal{Q})\rvert$	#
1	459,466	32	2,732	336	5
2	344,972	36	1	384	24
3	1,721	42	7	512	8
4	174,544	48	159	576	1
5	2	60	1	768	9
6	861	64	585	1,152	2
8	53,197	80	1	1,344	1
9	4	96	84	1,536	5
12	759	128	178	2,688	1
16	14,522	168	4	3,072	2
21	12	192	41	21,504	1
24	216	256	34	322,560	1

Table 2
Values of β

β	#	β	#
1	1,641	9	128,416
2	12,338	10	101,257
3	34,934	11	72,842
4	72,907	12	42,672
5	106,084	13	18,807
6	143,248	14	5,667
7	161,399	15	1,115
8	150,717	16	119

Table 3
Occurrences of derived triple systems

No	#	No	#	No	#	No	#
1	13,711	21	47,125	41	5,780	61	14,179
2	240,118	22	49,243	42	105	62	2,606
3	213,133	23	157,868	43	275	63	5,503
4	759,223	24	134,657	44	671	64	3,478
5	410,563	25	166,233	45	1,068	65	183
6	257,899	26	196,444	46	363	66	187
7	43,092	27	75,791	47	4,738	67	108
8	699,707	28	73,897	48	542	68	161
9	725,288	29	63,255	49	344	69	241
10	742,266	30	22,692	50	343	70	4,800
11	294,132	31	57,948	51	597	71	168
12	324,812	32	37,117	52	1,020	72	131
13	389,642	33	26,625	53	6,059	73	40
14	301,162	34	29,240	54	5,130	74	310
15	431,065	35	2,959	55	1,163	75	452
16	77,610	36	1,817	56	482	76	3,307
17	143,673	37	35	57	360	77	34
18	427,530	38	742	58	5,786	78	52
19	57,425	39	6,252	59	5,513	79	5
20	186,917	40	6,562	60	523	80	5

Table 4
Ranks

Rank	#
11	1
12	15
13	4,131
14	708,103
15	341,913